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FREE CONVECTION LIMITS IN THE OPEN THERMOSYPHON

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Abstract—Predictions are made of the lower limiting conditions of free convection in the vertical open thermosyphon of circular cross-section with uniform wall temperature, where the flow tends to zero and is superseded by conduction. The overall heat-transfer rate is then independent of tube length but proportional to radius, unless the length–radius ratio is below about 1·8, in which case it depends also on temperature conditions at the closed end. The corresponding Rayleigh number, which is a function of the Prandtl number only, is estimated for non-metallic fluids and for a liquid metal.

NOTATION

- a, tube radius;
- b, length–radius ratio of tube l/a;
- f, externally-produced acceleration;
- k, thermal conductivity;
- l, tube length;
- Nu_r, Nusselt number based on radius $= Q/2\pi lk(T_0 - T_1)$;
- Nu_r', modified Nusselt number defined in text;
- Nu_l, Nusselt number based on length $= Q/2\pi ak(T_0 - T_1) = bNu_r$;
- Pr, Prandtl number ν/α;
- Q, overall rate of heat transfer from the thermosyphon tube to the reservoir;
- R, distance from tube axis;
- r, dimensionless distance from tube axis, R/a;
- Ra_r, Rayleigh number based on radius $f\beta(T_0 - T_1)a^3/\nu\alpha$;
- T, temperature;
- T_0, temperature at walls;
- T_1, temperature on the tube axis at the top end;
- t, dimensionless temperature, $(T_0 - T)/(T_0 - T_1)$;
- t_1, radius-based Rayleigh number/length–radius ratio $= f\beta(T_0 - T_1)a^4/\nu\alpha l$;
- X, distance from top end of tube;
- x, dimensionless distance from open end, X/l.

Greek symbols

- α, thermal diffusivity;
- β, coefficient of cubic expansion;
- ν, kinematic viscosity.

1. INTRODUCTION

THE THEORETICAL and experimental researches of the last decade, stimulated by the projected applications of the free convection open thermosyphon to (a) gas-turbine rotor-blade cooling and (b) the extraction of heat from nuclear reactors, have resulted in the accumulation of a great deal of information concerning the several flow regimes and the unusual heat-transfer characteristics of the open thermosyphon. The geometry of this system is such that a heated tube, closed at its lower end, opens into a reservoir of cool fluid. The heated fluid rises over the inner walls of the tube through convective action to be replaced by a returning core of cool fluid from the reservoir.

The earlier investigations of Lighthill [1], Martin [2], and Hartnett and Welsh [3] were concerned with the study of fluids having Prandtl numbers greater than 0·6. More recent researches by Hartnett *et al.* [4], Leslie and Martin [5], Bayley *et al.* [6], and Bayley and Czekanski [7] have included the study of liquid metals whose Prandtl numbers are of order 0·01. These liquids are especially useful as heat-transfer media where high heat fluxes and high temperatures are involved. This later work has directed attention to the significance of conduction in the open thermosyphon, particularly in circumstances of high fluid thermal conductivity.

Conduction effects might also become predominant at small Rayleigh numbers, such that, under isothermal wall conditions and for $Pr \geqslant 0{\cdot}6$, t_1 is less than about 4000, the radius-based Nusselt number being less than 3. Figure 1 (reproduced from reference 5) shows that, for laminar flow, Nu_r then varies linearly with t_1, rather than according to the quarter-power relation, characteristic of boundary-layer flow, which otherwise prevails. Lighthill [1] has shown that in the linear range the boundary-layer has become so thick as to fill the tube; cool fluid thus mingles with this stream and becomes partly heated from the point of entry. This reduces the scale of the motion and the heat transfer. At values of t_1 marked by crosses in Fig. 1 as "similarity solutions" for three values of Pr, the local heat transfer falls linearly from a maximum at the orifice to zero at the closed end of the tube. The velocity and temperature profiles are "similar" at different sections along the tube axis. For still smaller t_1, flow extends down the tube to a section where heat transfer vanishes, below which the fluid is stagnant and at uniform temperature. As t_1 diminishes, the section referred to rises up the tube, so that the volume of stagnant fluid increases.

Figure 1 suggests that the linear relation between Nu_r and t_1 holds down to the point where both are zero, in which case the fluid would be stagnant throughout the tube. Another possibility suggests itself. If t_1 is small enough for convective motion to be restricted to the upper part of the tube, some heat may also be transferred from the walls through the stagnant fluid by conduction, such that the overall heat transfer differs little from that shown in Fig. 1 for $t_1 < 350$, where only convection is considered. Then, at a sufficiently small value of t_1 (or Ra_r, for a given tube) > 0, the layer of fluid in laminar flow at the orifice becomes infinitely thin, so that virtually all heat transfer is by conduction. This situation, which would yield the lowest Nu_r for convection, is the limiting case of zero flow. For smaller t_1, the relation between Nu_r and t_1 probably ceases to be linear because then the radial temperature distribution at the orifice is no longer governed by convection.

Predictions of Nu_r and Ra_r for the limiting case of zero flow form the subject of this paper. We therefore require a solution of the conduction equation for a cylinder whose outside wall is maintained at a uniform temperature T_0. This is greater than that of the axis, along which the temperature decreases to a minimum of T_1 at the orifice. In addition to the isothermal base condition assumed in previous investigations, where the closed end is at T_0, the effect of an adiabatic base is also considered. The thermal conductivity of the fluid is assumed to be uniform. The boundary condition at the open end is still determined by convective motion, even in the limiting case where away from the orifice there is no flow. While (as is here assumed to be the case) a linear relationship exists between Nu_r and t_1, Lighthill [1] has shown that, for $Pr \gg 1$,

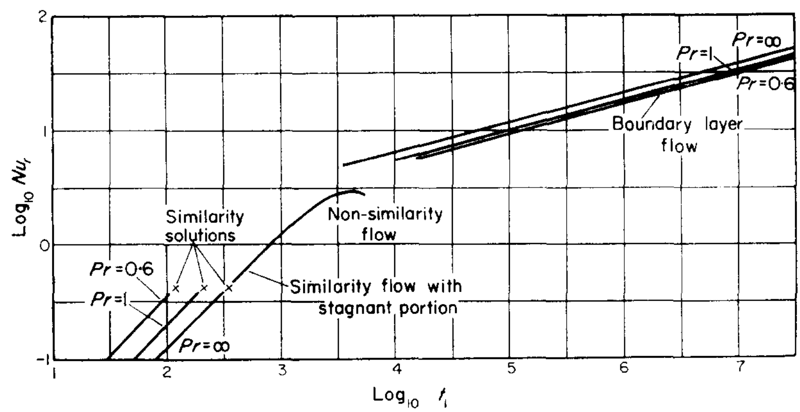

FIG. 1. Laminar flow regimes in the open thermosyphon for non-metallic fluids. [*J. Mech. Engng. Sci.* **1**, 184 (1959)].

the orifice radial temperature profile is that of the "similarity solution". (As derived by Leslie and Martin [5], this is given by equation (12) and illustrated in Fig. 3.) The implied insensitivity of the orifice radial temperature distribution to differing flow regimes within the tube, for $t_1 < 4000$, strengthens the justification for its use in the present case. The conduction equation is solved accordingly for Nu_r below, and the lower limit of Ra_r is subsequently estimated from knowledge of the linear relation between t_1 and Nu_r at low t_1.

2. HEAT CONDUCTION EQUATION

Cylindrical polar co-ordinates X and R are used, where the X-axis coincides with the axis of the cylinder shown in Fig. 2. X is measured

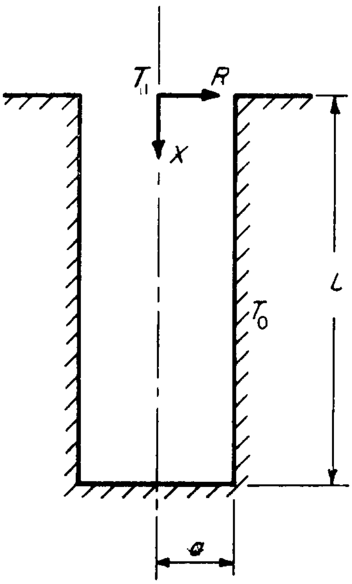

FIG. 2. Co-ordinate system for thermosyphon tube.

from an origin at the top, and R is the radial distance from the X-axis. For steady, axi-symmetric conditions, the heat conduction equation is

$$\frac{\partial^2 T}{\partial R^2} + \frac{1}{R}\frac{\partial T}{\partial R} + \frac{\partial^2 T}{\partial X^2} = 0 \qquad (1)$$

The boundary conditions are:

$$R = 0, \quad \frac{\partial T}{\partial R} = 0, \quad \text{for all } X \qquad (2)$$

$$R = 0, \quad X = 0, \quad T = T_1 \qquad (3)$$

$$R = a, \quad T = T_0, \quad \text{for all } X \qquad (4)$$

together with the orifice radial temperature profile for the convective laminar "similarity solution" specified by (12) below. If the base of the cylinder is adiabatic, we have also

$$X = l, \quad \frac{\partial T}{\partial X} = 0 \text{ for all } R \qquad (5)$$

but if the base is isothermal, as in previous work

$$X = l, \quad T = T_0 \text{ for all } R \qquad (6)$$

Conditions (5) and (6) will be seen below to lead to different results unless b is somewhat greater than unity.

The above equation and boundary conditions can be reduced to dimensionless form by the substitutions

$$X = lx, \quad R = ar, \quad T = T_0 - t(T_0 - T_1) \qquad (7)$$

Equation (1) then becomes

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{b^2}\frac{\partial^2 t}{\partial x^2} = 0 \qquad (8)$$

while the boundary conditions can be written

$$r = 0, \quad \frac{\partial t}{\partial r} = 0, \quad \text{for all } x \qquad (9)$$

$$r = 0, \quad x = 0, \quad t = 1 \qquad (10)$$

$$r = 1, \quad t = 0, \quad \text{for all } x \qquad (11)$$

$$t_{x=0} = 1 - 2{\cdot}394r^2 + 2{\cdot}632r^4 - 1{\cdot}667r^6 + 0{\cdot}430r^8 - \ldots \qquad (12)$$

(This profile is illustrated in Fig. 3.)

$$x = 1, \quad \frac{\partial t}{\partial x} = 0, \quad \text{for all } r \text{ (adiabatic base)} \qquad (13)$$

$$x = 1, \quad t = 0, \quad \text{for all } r \text{ (isothermal base)} \qquad (14)$$

The heat-conduction equation applied to the cylinder as a whole gives

$$Q = \int_0^l 2\pi ak\left(\frac{\partial T}{\partial R}\right)_{R=a} \mathrm{d}X + \int_0^a 2\pi Rk\left(\frac{\partial T}{\partial X}\right)_{X=l} \mathrm{d}R = \int_0^a 2\pi Rk\left(\frac{\partial T}{\partial X}\right)_{X=0} \mathrm{d}R \qquad (15)$$

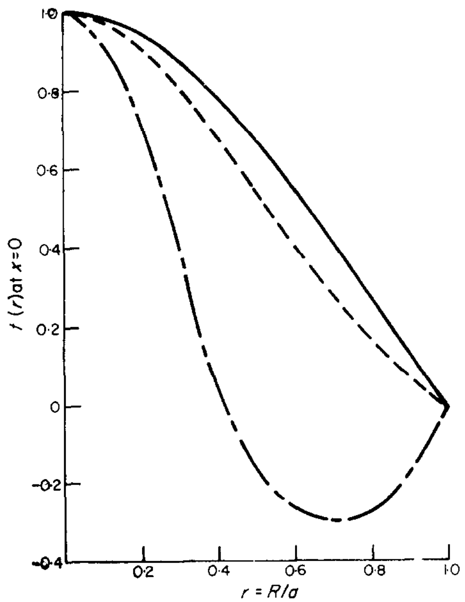

FIG. 3. Assumed radial temperature profile at orifice and relevant eigenfunctions.
———— First eigenfunction [$J_0(2{\cdot}404r)$]
— — — Second eigenfunction [$J_0(5{\cdot}520r)$]
- - - - - Convective laminar "similarity solution".

In dimensionless form this becomes

$$\int_0^1 \left(-\frac{\partial t}{\partial r}\right)_{r=1} \mathrm{d}x + \frac{1}{b^2}\int_0^1 \left(-\frac{\partial t}{\partial x}\right)_{x=1} r\,\mathrm{d}r = \frac{1}{b^2}\int_0^1 \left(-\frac{\partial t}{\partial x}\right)_{x=0} r\,\mathrm{d}r \qquad (16)$$

If the base is adiabatic, the second term on the L.H.S. of equation (16) is zero. The Nusselt number (based on radius, and wall area, corresponding to the heat entering the cylinder) is then given by

$$Nu_r = Qa/2\,\pi\, l\, ka(T_0 - T_1) = \frac{a}{(T_0 - T_1)\, l} \times \int_0^l \left(\frac{\partial T}{\partial R}\right)_{R=a} \mathrm{d}X = \int_0^1 \left(-\frac{\partial t}{\partial r}\right)_{r=1} \mathrm{d}x = \frac{1}{b^2}\int_0^1 \left(-\frac{\partial t}{\partial x}\right)_{x=0} r\,\mathrm{d}r \qquad (17)$$

If the base is isothermal, it is convenient to define a modified Nusselt number Nu_r' based on the heat *leaving* the cylinder, as follows:

$$Nu_r' = \frac{1}{b^2}\int_0^1 \left(-\frac{\partial t}{\partial x}\right)_{x=0} r\,\mathrm{d}r = \int_0^1 \left(-\frac{\partial t}{\partial r}\right)_{r=1} \mathrm{d}x + \frac{1}{b^2}\int_0^1 \left(-\frac{\partial t}{\partial x}\right)_{x=1} r\,\mathrm{d}r \qquad (18)$$

3. SOLUTION OF CONDUCTION EQUATION

Using the method of separating the variables described by Schneider [8], it can be shown that the general solution of equation (8) has the product form

$$t = [A\,\mathrm{e}^{hbx} + B\,\mathrm{e}^{-hbx}]\,[C\,J_0(hr) + D\,Y_0(hr)] \qquad (19)$$

where A, B, C and D are coefficients and $J_0(hr)$ and $Y_0(hr)$ are zero order Bessel functions of the first and second kinds respectively. These are defined by

$$J_0(hr) = 1 - \frac{(hr)^2}{2^2} + \frac{(hr)^4}{2^2.4^2} - \frac{(hr)^6}{2^2.4^2.6^2} + \ldots \qquad (20)$$

$$Y_0(hr) = J_0(hr)\log_e(hr) + \frac{(hr)^2}{2^2} - \frac{(hr)^4}{2^2.4^2}(1 + \tfrac{1}{2}) + \ldots \qquad (21)$$

At $r = 0$, t is finite for all x. But $Y_0(0) = -\infty$, and hence D must be zero. Then clearly $J_0(hr)$ satisfies condition (9) for all h. Applying boundary condition (10) to equation (19), with $D = 0$, we find that

$$A + B = 1 \qquad (22)$$

whence

$$t = [2A\sinh(hbx) + \mathrm{e}^{-hbx}]\,C\,J_0(hr) \qquad (23)$$

Since from condition (11), $r = 1$, $t = 0$, for all x, h must be such that $J_0(h) = 0$. Reference to tables of Bessel functions shows that h has the following eigenvalues: 2·404, 5·520, 8·653, 11·791, 14·930, . . . Thus a general solution of equation (23) is

$$t = \sum_{h=2\cdot404}^{h=\infty} [2\,A\sinh(hbx) + \mathrm{e}^{-hbx}]\,C\,J_0(hr) \qquad (24)$$

For the lowest eigenvalue, the eigenfunction $J_0(hr)$ is monotonic for $0 < r < 1$. Larger eigenvalues yield radial profiles having waves in them which increase in number with the eigenvalue. But any specified radial temperature distribution, such as equation (12), can be accommodated by assigning appropriate values to C in successive terms of the infinite series (24). Boundary condition (12) is in fact reproduced to an accuracy within 5 per cent by considering only the first two terms of the series solution (24), such that

$$h = 2{\cdot}404, \quad C = 0{\cdot}83; \quad h = 5{\cdot}520, \quad C = 0{\cdot}17;$$
$$h = 8{\cdot}653 \ldots \infty, \quad C = 0. \qquad (25)$$

The eigenfunctions $J_0(2{\cdot}404r)$ and $J_0(5{\cdot}520r)$ are shown in Fig. 3.

It remains to evaluate A according to whether (13) or (14) is to be satisfied. For an adiabatic base we have, from equation (23),

$$A = \mathrm{e}^{-hb}/2 \cos \mathbf{h}(hb) \qquad (26)$$

whence

$$Nu_r = \frac{1}{b}[0{\cdot}43 \tan \mathbf{h}(2{\cdot}404b) - 0{\cdot}058 \tan \mathbf{h}(5{\cdot}520b)] \qquad (27)$$

For the isothermal base, equations (14) and (23) give

$$A = -\, \mathrm{e}^{-hb}/2 \sin \mathbf{h}(hb) \qquad (28)$$

whence

$$Nu_r' = \frac{1}{b}[0{\cdot}43 \cot \mathbf{h}(2{\cdot}404b) - 0{\cdot}058 \cot \mathbf{h}(5{\cdot}520b)] \qquad (29)$$

4. LIMITING CONDITIONS FOR CONVECTION

Before discussing values of t_1 associated with equations (27) and (29) for the limiting case of zero flow, it is helpful to compare the axial temperature profiles (obtained by writing $J_0(hr) = 1$ in (24), in conjunction with (25), (26) and (28)) and the heat transfer for the two base boundary conditions with each other, and also with those of laminar flow.

For the temperature profiles this is done in Fig. 4, where $b = 1$. The laminar flow profile is that of the "similarity solution", where there is no stagnant fluid. The "no-flow" profiles diverge as x increases, such that, for $b = 1$, there is an adiabatic base temperature variation of up to $0{\cdot}15(T_0 - T_1)$. The variation is inversely related to b for b less than about 1·8. For larger b the "no-flow" profiles merge into a single distribution which is probably valid for any intermediate base boundary condition. Comparison

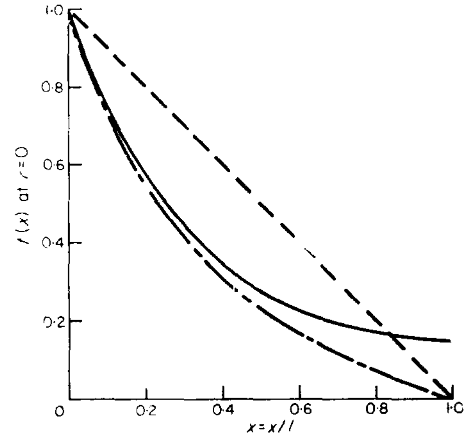

FIG. 4. Axial temperature profiles for unit length–radius ratio.
——— Axial zero-flow profile with adiabatic base and $b = 1$.
— - — Axial zero-flow profile with isothermal base and $b = 1$.
- - - - Convective laminar "similarity solution".

with the linear laminar profile suggests that correlation will be better where (a) there is stagnant fluid, as assumed in the analysis, so that $t = 0$ corresponds to $x < 1$, (b) b is large, which brings the "no-flow" profiles nearer the axes. The predictions made below for Ra_r when flow ceases may therefore be less reliable at small b.

Figure 5 shows the same comparisons for the heat transfer. As already implied, for $b > 1{\cdot}8$, overall zero-flow heat transfer is insensitive to base condition within the isothermal and adiabatic limits. Then, for $b > 1{\cdot}8$, equations (27) and (29) reduce to

$$Nu_r = \frac{Q}{2\,\pi\, lk(T_0 - T_1)} = \frac{0{\cdot}372a}{l} \qquad (30)$$

where the length-based Nusselt number

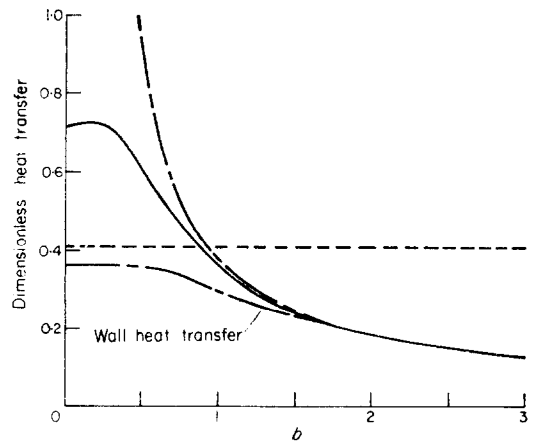

FIG. 5. Zero-flow heat transfer as a function of length–radius ratio.
———— Adiabatic base.
—— – —— Isothermal base.
– – – – – – – Convective laminar "similarity solution".

$Nu_l = bNu_r$ is constant, and the overall heat transfer Q is proportional to tube radius, but independent of length. The hyperbolic tangent and cotangent terms in (27) and (29), which tend rapidly to unity as b increases, make it clear that these remarks hold for any specified orifice boundary condition which can be accommodated by equation (24).

Equation (30) is obviously a good approximation for either base boundary condition for b down to 0·9, where the zero-flow heat transfer coincides with that of the laminar "similarity solution". For lower b it accurately represents the limiting heat transfer for convection only for some intermediate base condition. The isothermal base contributes increasingly to the overall heat transfer as b diminishes below 1·8. Meanwhile the wall heat transfer attains a value of 0·358 at $b = 0{\cdot}5$ and is nearly constant thereafter. With an adiabatic base, Nu_r reaches a maximum of 0·725 at $b = 0{\cdot}2$, falling to 0·714 in the limiting case of $b = 0$.

$Nu_r = 0{\cdot}407$, shown in Fig. 5 for the laminar "similarity solution", corresponds to the following approximate relation between t_1 and Pr, due to Leslie and Martin [5], which has been experimentally confirmed [2]:

$$t_1 + \frac{135{\cdot}5}{Pr} = 345{\cdot}5 \text{ for } 0{\cdot}4 < Pr < \infty \quad (31)$$

The linear variation of Nu_r with t_1 for $Nu_r < 3$ is then given by

$$Nu_r = \frac{t_1}{850 - 333/Pr} \quad (32)$$

The lower limiting value of Ra_r for convection, where the same heat transfer arises from zero flow, is obtained by combining (30) with (32), whence

$$Ra_r = bt_1 = 316 - \frac{124}{Pr} \quad (33)$$

which, as mentioned above, is of uncertain accuracy for small b.

For mercury, the experiments of Bayley and Czekanski [7] have shown that, where $2 < b < 12$, the linear relation between Nu_r and t_1 takes the form

$$Nu_r = 0{\cdot}0011\, t_1\, Pr \text{ for } t_1\, Pr < \frac{7060}{b^{0{\cdot}643}} \quad (34)$$

The orifice radial temperature profile under these conditions was not reported. If it be assumed the same as for conventional fluids, i.e. according to (12), the criterion comparable with (33) is, from (30) and (34)

$$Ra_r Pr = 338 \quad (35)$$

The values of Ra_r derived from (33) and (35) are lower than those so far reached in experimental investigations (where no departure from a linear relation between Nu_r and t_1 has been observed), but, since $Ra_r \alpha a^3$, there is no reason why they should not be attained in tubes of sufficiently small diameter.

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REFERENCES

1. M. J. LIGHTHILL, Theoretical considerations on free convection in tubes, *Quart. J. Mech.* **6**, 398 (1953).
2. B. W. MARTIN, Free convection in an open thermosyphon, with special reference to turbulent flow, *Proc. Roy. Soc.* **A 230**, 502 (1955).
3. J. P. HARTNETT and W. E. WELSH, Experimental studies of free convection heat transfer in a vertical tube with uniform wall heat flux, *Trans. Amer. Soc. Mech. Engrs* **79**, 1551 (1957).
4. J. P. HARTNETT, W. E. WELSH and F. W. LARSEN,

Free convection heat transfer to water and mercury in an enclosed cylindrical tube, *Chem. Engng Progr. Symposium Series, Nuclear Engng* **55**, 85 (1959).
5. F. M. LESLIE and B. W. MARTIN, Laminar flow in the open thermosyphon, with special reference to small Prandtl numbers, *J. Mech. Engng Sci.* **1**, 184 (1959).
6. F. J. BAYLEY, P. A. MILNE and D. E. STODDART, Heat transfer by free convection in a liquid metal, *Proc. Roy. Soc.* **A 265**, 97 (1961).
7. F. J. BAYLEY and J. CZEKANSKI, Experimental investigation of free convection in a liquid metal in long tubes, *J. Mech. Engng Sci.* **5**, 295 (1963).
8. P. J. SCHNEIDER, *Conduction Heat Transfer*. Addison-Wesley, Cambridge, Massachusetts (1955).

Résumé—Les conditions limites inférieures de convection naturelle dans le thermosiphon vertical ouvert sont prédites avec une température uniforme de la paroi où l'écoulement tend vers zéro et est remplacé par la conduction. Le flux global de transport de chaleur est alors indépendant de la longueur du tube mais proportionnel au rayon, à moins que le rapport longueur sur rayon soit en-dessous de 1,8 environ, dans quel cas il dépend aussi des conditions de température à l'extrémité fermée. Le nombre de Rayleigh correspondant, qui est une fonction seulement du nombre de Prandtl, est estimé pour des fluides non métalliques et pour un métal liquide.

Zusammenfassung—Für die unteren Grenzbedingungen bei freier Konvektion in einen senkrechten, oben offenen "Wärmesiphon", wo die Strömung nach Null geht und die Wärmeleitung ausschlaggebend wird, werden Vorhersagen gemacht. Der gesamte Wärmeübergang ist dann von der Rohrlänge unabhängig, aber proportional dem Radius, wenn das Verhältnis Länge zu Radius nicht unter 1,8 liegt. Darunter hängt er auch von den Temperaturverhältnissen am geschlossenen Rohrende ab. Die entsprechende Rayleighzahl, die nur eine Funktion der Prandtlzahl ist, wird für nichtmetallische Flüssigkeiten und für flüssige Metalle abgeschätzt.

Аннотация—Рассчитаны нижние предельные условия свободной конвекции в вертикальном открытом термосифоне с постоянной температурой стенки, когда конвекция стремится к нулю и замещается теплопроводностью. Общий тепловой поток в этом случае не зависит от длины трубы, но пропорционален радиусу, если отношение длины к радиусу не меньше 1,8. В последнем случае она зависит также от температурных условий на закрытом конце. Определено соответствующее число Релея для неметаллических жидкостей и жидких металлов, являющееся функцией только числа Прандтля.